A Complete Kinematic Analysis of the 3-RPS Parallel Manipulator

Josef Schadlbauer, Dominic R. Walter and Manfred L. Husty

Abstract

A 3-RPS manipulator is a three degree of freedom (DOF) parallel manipulator. It consists of a equilateral triangular fixed platform and a similar moving platform connected by 3 identical RPS (revolute-prismatic-spherical) legs. This manipulator has got a lot of attention in the literature. But as it turns out all of the papers are incomplete. This paper closes the gap and gives a complete description of the forward kinematics and of all operation modes of this manipulator, using *Study*'s kinematic mapping. For this purpose algebraic constraint equations for each RPS leg are derived. The constraint equations together with the *Study* equation and a normalization term determine an ideal, representing the complete algebraic and kinematic description of the manipulator. This ideal is tested for possible decomposition to reveal different operation modes and then the different corresponding systems of equations are solved. Furthermore all singular poses and transitions between the operation modes are computed using the Jacobian of the aforementioned system.

Keywords: 3-RPS-manipulator, Direct Kinematics, Working Modes, Singularities

1 Introduction

A 3-RPS manipulator is a three degree of freedom (DOF) parallel manipulator. It consists of a equilateral triangular fixed platform and a similar moving platform connected by 3 identical RPS legs, where the first joint (R-joint) is connected to the base and the last joint (S-joint) is connected to the moving platform (see Fig. 1). The legs are extensible, their lengths are changed with prismatic joints (P-joints), thereby moving the platform with three DOFs. In the past few years the 3-RPS got a lot of attention in literature, see e.g. [1]. In [2] an overview of the existing results can be found and especially it is stated that Hunt (1983) has introduced this type of lower degree of parallel manipulator. In [2] Gallardo et al. present a kinematic analysis of the manipulator including position, velocity and acceleration behavior using vector loop equations for the position analysis and screw theory for velocity and acceleration analysis. Huang et al. [3] did an analysis on different states of this manipulator, especially when the platform is parallel to the base also using screw theory. Already Tsai [1] had the correct number of solutions of the direct kinematics, but as it turned out, due to the applied local methods (also in [2]), a complete description of working modes and singular poses

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was overlooked and is still missing. This gap was closed by Basu and Ghosal [4], who gave a characterization of special singular poses of the manipulator.

In this paper, using an algebraic description of the manipulator, together with *Study*'s kinematic mapping, a complete characterization of the the forward kinematics, the operation modes, the singular poses and the transitions between the operation modes will be given. The paper is organized as follows: In Section 2 a description of the architecture of the 3-RPS is given. The derivation of constraint equations is done in Section 3. The different operation modes and singularities are discussed in Sections 4 and 5. In Section 6 a complete description of all poses is given where changing from one operation mode to the other is possible. Finally, in Section 7 the different operation modes are interpreted geometrically and an example which shows a transition between the operation modes is introduced.

2 Robot Design

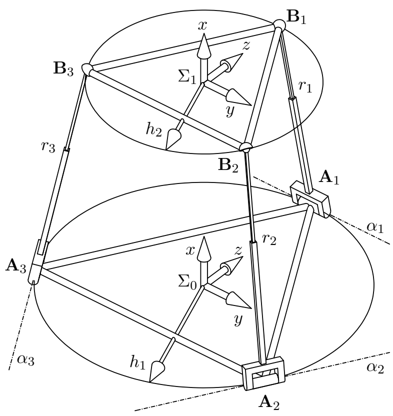

Figure 1: Design of the 3-RPS parallel robot

With respect to Fig. 1 we consider the 3-RPS parallel manipulator with the following architecture: The base of the 3-RPS consists of an equilateral triangle with vertices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ and circumradius h_1. The origin of the fixed frame Σ_0 coincides with the circumcenter of the triangle $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$. The yz-plane of Σ_0 is defined by the plane $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$. Finally, $\mathbf{A}_1$ lies on the z-axis of Σ_0. In the platform there is another equilateral triangle with vertices $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ and circumradius h_2. The circumcenter of the triangle $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ lies in the origin of Σ_1, which is the moving frame. Again, the plane defined by $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ coincides with the yz-plane of Σ_1 and $\mathbf{B}_1$ lies on the z-axis of Σ_1. The two design parameters h_1 and h_2 are taken to be strictly positive numbers. Now each pair of vertices $\mathbf{A}_i$, $\mathbf{B}_i$ ($i = 1, \ldots, 3$) is connected by a limb, with a rotational joint at $\mathbf{A}_i$ and a spherical joint at $\mathbf{B}_i$. The length of each limb is denoted by r_i and is adjusted via an actuated prismatic joint. The axes α_i of the rotational joints at $\mathbf{A}_i$ are tangent to the circumcircle and therefore lie within the

yz-plane of Σ_0.

Overall we have five parameters, namely h_1, h_2, r_1, r_2 and r_3. While h_1 and h_2 determine the design of the manipulator, the parameters r_1, r_2 and r_3 are joint parameters, which determine the motion of the robot. We can consider the joint parameters to be like design parameters when they are assigned with specific leg lengths r_i. From this point of view we will discuss the direct kinematics of the manipulator and furthermore we will assume that all parameters are strictly positive real numbers.

3 Derivation of the Constraint Equations

Deriving the constraint equations is one essential step in solving the direct kinematics. To compute these equations which describe all possible solutions of the direct kinematics, i.e. all possible poses of Σ_1 with respect to Σ_0 and with that all poses of the platform, we use the *Study*-parameterization of the motion group $SE(3)$. The vertices of the base triangle and the platform triangle in Σ_0 resp. Σ_1 are

$$\mathbf{A}_1 = (1, 0, 0, h_1), \quad \mathbf{A}_2 = (1, 0, \sqrt{3}h_1/2, -h_1/2), \quad \mathbf{A}_3 = (1, 0, -\sqrt{3}h_1/2, -h_1/2)$$
$$\mathbf{b}_1 = (1, 0, 0, h_2), \quad \mathbf{b}_2 = (1, 0, \sqrt{3}h_2/2, -h_2/2), \quad \mathbf{b}_3 = (1, 0, -\sqrt{3}h_2/2, -h_2/2)$$

thereby using projective coordinates with the homogenizing coordinate in first place. To avoid confusion coordinates with respect to Σ_0 are written in capital letters and those with respect to Σ_1 are in lower case.

To obtain the coordinates $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ of $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$with respect to Σ_0 a coordinate transformation has to be applied. To describe this coordinate transformation we use *Study's* parameterization of a spatial Euclidean transformation matrix $\mathbf{M} \in SE(3)$ (for detailed information on this approach see [5]).

$$\mathbf{M} = \begin{pmatrix} x_0^2 + x_1^2 + x_2^2 + x_3^2 & \mathbf{0}^\top \\ \mathbf{M}_T & \mathbf{M}_R \end{pmatrix}, \quad \mathbf{M}_T = \begin{pmatrix} 2(-x_0y_1 + x_1y_0 - x_2y_3 + x_3y_2) \\ 2(-x_0y_2 + x_1y_3 + x_2y_0 - x_3y_1) \\ 2(-x_0y_3 - x_1y_2 + x_2y_1 + x_3y_0) \end{pmatrix}$$

$$\mathbf{M}_R = \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_1x_3 + x_0x_2) \\ 2(x_1x_2 + x_0x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_2x_3 + x_0x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}$$

The vector $\mathbf{M}_T$ represents the translational part and $\mathbf{M}_R$ represents the rotational part of the transformation $\mathbf{M}$. The parameters $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3$ which appear in the matrix $\mathbf{M}$ are called *Study*-parameters of the transformation $\mathbf{M}$. The mapping

$$\kappa\colon SE(3) \to P \in \mathbb{P}^7 \tag{1}$$
$$\mathbf{M}(x_i, y_i) \mapsto (x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3)^T \neq (0:0:0:0:0:0:0:0)^T$$

is called *kinematic mapping* and maps each Euclidean displacement of $SE(3)$ to a point P on $S_6^2 \subset \mathbb{P}^7$. In this way, every projective point $(x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3) \in \mathbb{P}^7$ represents a spatial Euclidean transformation, if it fulfills the following equation and inequality:

$$x_0y_0 + x_0y_0 + x_0y_0 + x_0y_0 = 0, \qquad x_0^2 + x_1^2 + x_2^2 + x_3^2 \neq 0. \tag{2}$$

All points which fulfill (2, left) lie on a 6-dimensional quadric $S_6^2 \subset \mathbb{P}^7$, the so called *Study*-quadric. Eq. (2, right) is a normalization term and points with $x_0 = x_1 = x_2 = x_3 = 0$ do not represent a Euclidean transformation, they form the 3-dimensional *exceptional generator* contained in S_6^2. The coordinates of $\mathbf{b}_i$ with respect to Σ_0 are obtained by:

$$\mathbf{B}_i = \mathbf{M} \cdot \mathbf{b}_i, \quad i = 1, \dots, 3.$$

Now, as the coordinates of all vertices are given in terms of the transformation parameters x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3 and the design constants, we obtain constraint equations by examining the geometry of the manipulator more closely. First of all the limb connecting $\mathbf{A}_i$ and $\mathbf{B}_i$ has to be orthogonal to the corresponding rotational axis α_i. That means, the scalar product of the vector connecting $\mathbf{A}_i\mathbf{B}_i$ and the direction of α_i vanishes. After computing this product and removing the common denominator $(x_0^2 + x_1^2 + x_2^2 + x_3^2)$ the following equations are obtained:

$$\begin{aligned}
\widetilde{g}_1 &: x_0y_2 - x_1y_3 - x_2y_0 + x_3y_1 - h_2x_2x_3 + h_2x_0x_1 = 0 \\
\widetilde{g}_2 &: 4\sqrt{3}h_2x_0x_1 + 2\sqrt{3}h_2x_2x_3 - 2\sqrt{3}x_0y_2 + 2\sqrt{3}x_1y_3 + 2\sqrt{3}x_2y_0 - 2\sqrt{3}x_3y_1 \\
&\quad +3h_2x_2^2 - 3h_2x_3^2 - 6x_0y_3 - 6x_1y_2 + 6x_2y_1 + 6x_3y_0 = 0 \\
\widetilde{g}_3 &: 4\sqrt{3}h_2x_0x_1 + 2\sqrt{3}h_2x_2x_3 - 2\sqrt{3}x_0y_2 + 2\sqrt{3}x_1y_3 + 2\sqrt{3}x_2y_0 - 2\sqrt{3}x_3y_1 \\
&\quad -3h_2x_2^2 + 3h_2x_3^2 + 6x_0y_3 + 6x_1y_2 - 6x_2y_1 - 6x_3y_0 = 0,
\end{aligned}$$

which after some elementary manipulations simplify to:

$$\begin{aligned}
g_1 &: x_0x_1 = 0 \\
g_2 &: h_2x_2^2 - h_2x_3^2 - 2x_0y_3 - 2x_1y_2 + 2x_2y_1 + 2x_3y_0 = 0 \\
g_3 &: 2h_2x_0x_1 + h_2x_2x_3 - x_0y_2 + x_1y_3 + x_2y_0 - x_3y_1 = 0.
\end{aligned} \tag{3}$$

Next we make use of the limb lengths. In the direct kinematics the joint parameters are given, therefore the distance between $\mathbf{A}_i$ and $\mathbf{B}_i$ has to be $r_i = const$ and from this follows that $\mathbf{B}_i$ has the freedom to move along a circle with center $\mathbf{A}_i$, which lies in a plane perpendicular to α_i. The constraint equation for this distance property has been derived in [6] for the direct kinematics of the general 6-SPS-Stewart-Gough-platform. Applying this formula for the design parameters at hand we obtain:

$$\begin{aligned}
g_4 :&(h_1 - h_2)^2x_0^2 + (h_1 + h_2)^2x_1^2 + (h_1 + h_2)^2x_2^2 + (h_1 - h_2)^2x_3^2 \\
&+ 4(h_1 - h_2)x_0y_3 + 4(h_1 + h_2)x_1y_2 - 4(h_1 + h_2)x_2y_1 \\
&- 4(h_1 - h_2)x_3y_0 + 4(y_0^2 + y_1^2 + y_2^2 + y_3^2) - (x_0^2 + x_1^2 + x_2^2 + x_3^2)r_1^2 = 0
\end{aligned}$$

$$\begin{aligned}
g_5 :&(h_1 - h_2)^2x_0^2 + (h_1 + h_2)^2x_1^2 + (h_1^2 + h_2^2 - h_1h_2)x_2^2 + (h_1^2 + h_2^2 + h_1h_2)x_3^2 - 2(h_1 \\
&- h_2)x_0y_3 - 2(h_1 + h_2)x_1y_2 + 2(h_1 + h_2)x_2y_1 + 2(h_1 - h_2)x_3y_0 - 2\sqrt{3}(h_1 \\
&- h_2)x_0y_2 + 2\sqrt{3}(h_1 + h_2)x_1y_3 + 2\sqrt{3}(h_1 - h_2)x_2y_0 - 2\sqrt{3}(h_1 + h_2)x_3y_1 \\
&- 2\sqrt{3}h_1h_2x_2x_3 + 4(y_0^2 + y_1^2 + y_2^2 + y_3^2) - (x_0^2 + x_1^2 + x_2^2 + x_3^2)r_2^2 = 0 \\
g_6 :&(h_1 - h_2)^2x_0^2 + (h_1 + h_2)^2x_1^2 + (h_1^2 + h_2^2 - h_1h_2)x_2^2 + (h_1^2 + h_2^2 + h_1h_2)x_3^2 - 2(h_1 \\
&- h_2)x_0y_3 - 2(h_1 + h_2)x_1y_2 + 2(h_1 + h_2)x_2y_1 + 2(h_1 - h_2)x_3y_0 + 2\sqrt{3}(h_1 \\
&- h_2)x_0y_2 - 2\sqrt{3}(h_1 + h_2)x_1y_3 - 2\sqrt{3}(h_1 - h_2)x_2y_0 + 2\sqrt{3}(h_1 + h_2)x_3y_1 \\
&+ 2\sqrt{3}h_1h_2x_2x_3 + 4(y_0^2 + y_1^2 + y_2^2 + y_3^2) - (x_0^2 + x_1^2 + x_2^2 + x_3^2)r_3^2 = 0.
\end{aligned}$$

To complete the system, we add the Study-equation (2), because all the solutions have to be within the Study-Quadric.

$$g_7 : x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0 \tag{4}$$

Now we have to find all points in $\mathbb{P}^7$, which fulfill these seven equations under the condition $x_0^2 + x_1^2 + x_2^2 + x_3^2 \neq 0$. By solving this system of equations, we get all points corresponding to all possible poses of the platform for a given set of joint parameters r_1, r_2, r_3. As the coordinates in $\mathbb{P}^7$ are homogeneous there is still a freedom to normalize the coordinates. To exclude the exceptional generator we add the normalization equation: $g_8 : x_0^2 + x_1^2 + x_2^2 + x_3^2 - 1 = 0$. This equation ensures, that no point of the exceptional generator appears as a solution. But now we have to keep in mind that for every projective solution we get two affine representatives. This has to be taken into account in counting the number of solutions in the following sections.

4 Solving the System

Now we want to study the system of equations $\{g_1, \ldots, g_8\}$ using methods of algebraic geometry (see e.g. [7] for basics of algebraic geometry). Without loss of generality the system is simplified by scaling $h_1 = 1$. To deal with the equations we will consider the following ideal

$$\mathcal{I} = \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8 \rangle ,$$

where g_i denotes the polynomial on the left-hand side of the corresponding equation. First of all the following sub-ideal is examined, which is independent of the joint parameters r_1, r_2 and r_3

$$\mathcal{J} = \langle g_1, g_2, g_3, g_7 \rangle .$$

To determine the operation modes of this manipulator a primary decomposition of this ideal is computed. As it turns out, the ideal $\mathcal{J}$ decomposes:

$$\mathcal{J} = \bigcap_{i=1}^{3} \mathcal{J}_i \tag{5}$$

with

$$\begin{aligned}
\mathcal{J}_1 = \big\langle x_0, x_1y_1 + x_2y_2 + x_3y_3, h_2x_2^2 - h_2x_3^2 - 2x_1y_2 + 2x_2y_1 + 2x_3y_0, \\
h_2x_2x_3 + x_1y_3 + x_2y_0 - x_3y_1 \big\rangle \\
\mathcal{J}_2 = \big\langle x_1, x_0y_0 + x_2y_2 + x_3y_3, h_2x_2^2 - h_2x_3^2 - 2x_0y_3 + 2x_2y_1 + 2x_3y_0, \\
h_2x_2x_3 - x_0y_2 + x_2y_0 - x_3y_1 \big\rangle \\
\mathcal{J}_3 = \langle x_0, x_1, x_2, x_3 \rangle .
\end{aligned} \tag{6}$$

From this splitting one can already conclude that the workspace of this manipulator splits into three components. By inspection of the vanishing set $\mathcal{V}(\mathcal{J}_3 \cup g_8)$ one can conclude that it is empty, because its ideal contains the polynomials $\{x_0, x_1, x_2, x_3, x_0^2 + x_1^1 + x_2^2 + x_3^2 - 1\}$ which never vanish simultaniously. So there are only two components (or operation modes) left. To complete the analysis we add the remaining equations g_4, g_5, g_6 and g_8 by writing

$$\mathcal{K}_i := \mathcal{J}_i \cup \langle g_4, g_5, g_6, g_8 \rangle .$$

The vanishing set of the ideal $\mathcal{I}$ can be described as:

$$\mathcal{V}(\mathcal{I}) = \bigcup_{i=1}^{3} \mathcal{V}(\mathcal{K}_i). \tag{7}$$

4.1 Solutions for arbitrary design parameters

First of all we describe of the solution for arbitrary design parameters, assuming that the 5 parameters (h_i, r_j) are generic. To obtain the number of solutions for the direct kinematics the Hilbert dimension for both $\mathcal{K}_1$ and $\mathcal{K}_2$ is computed with $h_1 = 1$. It turns out that

$$dim(\mathcal{K}_i) = 0, \qquad i = 1, 2, \tag{8}$$

which means that there are finitely many solutions for the direct kinematics in each component $\mathcal{K}_i$, or finitely many assembly modes. In this case $dim(\mathcal{K}_i)$ represents the dimension over $\mathbb{C}[h_1, h_2, r_1, r_2, r_3]$. Because it was not possible to determine the dimension for a general h_2, we substituted randomly chosen rational numbers and calculated the dimension. To obtain the number of solutions and the solutions themselves, a univariate polynomial was computed. This was done using the algebraic manipulation software Singular. We obtain a degree eight univariate polynomial for each component $\mathcal{K}_1$ and $\mathcal{K}_2$. The degree of this univariate polynomial has to be halved (see Section 3):

$$|\mathcal{V}(\mathcal{K}_i)| = 4, \qquad i = 1, 2. \tag{9}$$

Overall we have therefore 8 solutions for the direct kinematics in agreement with [1, 2], when the design parameters are chosen arbitrarily.

4.2 Solutions for equal leg lengths

In this section we are going to take a closer look into the case with equal leg lengths

$$r_1 = r_2 = r_3 = r. \tag{10}$$

In this case, the Hilbert dimension is computed quite easily and it follows that

$$\dim(\mathcal{K}_i) = 0, \qquad i = 1, 2, \tag{11}$$

what means that there are still finitely many solutions in each component. We obtain

$$|\mathcal{V}(\mathcal{K}_i)| = 2, \qquad i = 1, 2. \tag{12}$$

So we have 4 solutions in case of equal leg lengths. The home position, which can be seen in Fig. 1 belongs to the case of equal limb lengths. The solution is:

$$x_0 = 1, x_1 = 0, x_2 = 0, x_3 = 0, y_0 = 0, y_1 = -\sqrt{r^2 - (h_1 - h_2)^2}, y_2 = 0, y_3 = 0.$$

5 Singular Poses of the Manipulator

In this section we want to describe all singular poses of the 3-RPS parallel manipulator. r_1, r_2 and r_3 have been fixed in the sections before, but now we want to treat these parameters as variables, which can change as to move the manipulator. We found out that

$$\overline{dim}(\mathcal{K}_i) = 3, \qquad i = 1, 2, \tag{13}$$

what is clear, because the 3-RPS is a 3-DOF parallel manipulator. In this case $\overline{dim}(\mathcal{K}_i)$ denotes the dimension over $\mathbb{C}[h_1, h_2]$, but we have to keep in mind that $h_1 = 1$, while h_2 was chosen general. In the kinematic image space the singular poses of both components are computed by taking the Jacobian J_i of each system of polynomials $\mathcal{K}_i$ and computing the determinant $S_i : \det(J_i) = 0$. This results in a hyper-variety of degree 8 in each component.

$$S_1 : x_1 \cdot p^7(x_2, x_3, y_0, y_1, y_2, y_3) = 0 \text{ and } S_2 : x_0 \cdot p^7(x_2, x_3, y_0, y_1, y_2, y_3) = 0 \tag{14}$$

It is obvious from Eq. (14), that the linear space $x_0 = x_1 = 0$ belongs to the singularity variety. The remaining parts in each component are varieties of degree seven.

It is desirable to have the singularity set also in the joint space. To obtain this set we define a map

$$\iota : S_i \to \mathcal{S}_i \in \mathcal{R}, \qquad i = 1, 2,$$

where $\mathcal{R}$ denotes the 3-dimensional joint space r_1, r_2, r_3. The singularity set consists of surfaces in $\mathcal{R}$. It was not possible to compute the singularity surfaces in general, but after assigning a value to h_2 the computation could be done. For the following examples we assign $h_2 = 2$. To compute the singularity set of the manipulator in the joint space $\mathcal{R}$, the determinant of the *Jacobian* of the system $\mathcal{I}$ has to be added. Then $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3$ have to be eliminated to obtain a polynomial in r_1, r_2, r_3 only. This algorithm was performed for each system $\mathcal{K}_i$ separately.

For the system $\mathcal{K}_1$ and the *Jacobian* we have $x_0 = 0$. Eliminating the remaining variables $x_1, x_2, x_3, y_0, y_1, y_2, y_3$ leads to a degree 32 polynomial in r_1, r_2, r_3, which factorizes in a product of a degree 24 and a degree 8 polynomial. The corresponding degree 24 surface is shown in Fig. 2 and Fig. 3 shows the degree 8 surface. A similar situation occurs for the system $\mathcal{K}_2$. One obtains again a polynomial of degree 32, that factorizes in a product of a degree 24 and a degree 8 polynomial. The corresponding singularity surface of degree 24 for this component is shown in Fig. 4. The remaining degree 8 surface is the same as in the first case. An interpretation of the common degree 8 factor is given in the next section. The surfaces in Figs. 2-4 are plotted in the complete 3-dimensional joint space, mechanically relevant are only those parts which lie in the first octant of the coordinate system where all r_i are positive.

6 Changing the Operation Modes

As we have seen in the previous sections, the workspace of the 3-RPS decomposes into two operation modes. To change from one operation mode to the other, the manipulator has to take a pose which belongs to both systems. The solution lies in the set $\mathcal{V}(\mathcal{K}_1 \cup \mathcal{K}_2)$ ($x_0 = x_1 = 0$). Furthermore this pose has to be a singular pose of the

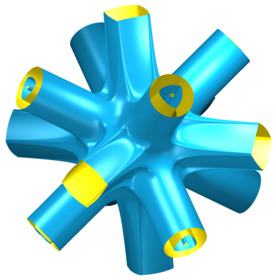

Figure 2: Singularity surface of $\mathcal{K}_1$

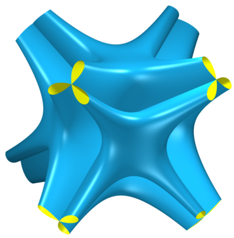

Figure 3: Singularity surface of $\mathcal{K}_1 \cup \mathcal{K}_2$ (changing modes)

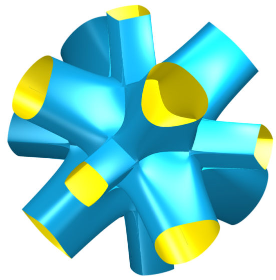

Figure 4: Singularity surface of $\mathcal{K}_2$

manipulator. In Section 5 the singularity surfaces in the joint-space were calculated and the commmon degree 8 polynomial (15) turned out to be solution for the changing of operation modes. The dimension of the intersection is $\overline{dim}(\mathcal{K}_1 \cup \mathcal{K}_2) = 2$. The condition on the limb lengths r_i to change the operation mode is ($h_1 = 1$ and $h_2 = 2$):

$$\begin{aligned} & r_1^8+r_2^8+r_3^8-2r_1^6(12+r_2^2+r_3^2)-2r_2^6(12+r_1^2+r_3^2)-2r_3^6(12+r_1^2+r_2^2)+3r_1^2r_2^2(r_1^2r_2^2 \\ & +12r_1^2+12r_2^2)+3r_1^2r_3^2(r_1^2r_3^2+12r_1^2+12r_3^2)+3r_2^2r_3^2(r_2^2r_3^2+12r_2^2+12r_3^2)-144r_1^2r_2^2r_3^2 = 0. \end{aligned} \tag{15}$$

7 Interpretation and Examples

In this section examples will show poses corresponding to different operation modes of the manipulator. For this purpose we chose the design parameters to be $h_1 = 1$ and $h_2 = 2$. The degree 8 singularity surface for this case was computed in Section 6 (Eq. (15)). Now a geometrical interpretation of the two different operation modes will be given. To do so, a closer examination of the screw axes of the transformations will be necessary. The direction of the screw axis is determined by x_1, x_2 and x_3 and the angle by x_0. If we take $a = (a_1, a_2, a_3)^T$ to be the normed direction of this screw axis, then (compare *Euler*-parameters of a rotation) $x_0 = \cos(\varphi/2)$ and $(x_1, x_2, x_3)^T = \sin(\varphi/2) \cdot (a_1, a_2, a_3)^T$ and φ is the angle of rotation.

To show a transition from one operation mode to the other we proceed as follows: two leg lengths are fixed as $r_1 = 6, r_2 = 5$, while the third one r_3 is calculated such that the manipulator is in a singular pose. To obtain a singular pose the degree 8 singularity condition (15) has to be solved for r_3; one numerical solution is $r_3 \sim 5.402$. Now the leg length r_3 is assumed to be a parameter, inducing a one parameter motion of the platform. During the motion the manipulator will be in a singular pose belonging to both components when it reaches the parameter value $r_3 \sim 5.402$ computed above.

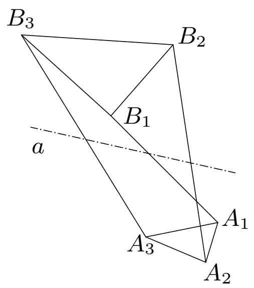

Figure 5: General pose in operation mode $x_0 = 0$ with screw axis a

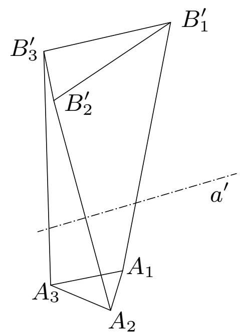

Figure 6: General pose in operation mode $x_1 = 0$ with screw axis a'

7.1 Operation mode $x_0 = 0$

First of all we will give an interpretation of the operation mode in which $x_0 = 0$. In this case we have $\varphi = \pi$. This means that the transformation is (as Study called it) a π-screw, i.e. a rotation about an axis a with angle π and a translation in the direction a. So all possible poses of the manipulator in this operation mode are obtained by transforming the platform from the identity where the coordinate systems Σ_0 and Σ_1 coincide via a π-screw. A possible pose and the screw axis a of this operation mode can be seen in Fig. 5 (leg lengths: $r_1 = 6, r_2 = 5$ and $r_3 = 5.402 + 3/4$).

7.2 Operation mode $x_1 = 0$

In this case the positions of the screw axes are special. Regarding to Fig. 1 the x-component of the screw axes is 0, so all screw axes a' are parallel to the yz-plane. A possible pose and the screw axis a' can be seen in Fig. 6. In this case the leg lengths were chosen to be $r_1 = 6, r_2 = 5$ and $r_3 = 5.402 + 3/4$.

7.3 Crossing the singularity $x_0 = x_1 = 0$

To get from one operation mode to the other, one has to cross a singularity and also $x_0 = x_1 = 0$ has to be fulfilled. The screw axis of the transformation has to be parallel to the yz-plane and the angle has to be π. Now there are two possibilities to get out of this singular pose, namely the operation mode $x_0 = 0$, in which the transformation is a π-screw, and the operation mode with $x_1 = 0$, in which the screw axis stays parallel to the yz-plane. In Fig. 7 two instants of such a motion out of a singular pose are shown. To get out of the singularity, at least one leg length has to be changed. In this example r_3 is extended step by step and the system is solved again. In the singular starting pose, which is displayed in the leftmost picture of Fig. 7 the two solutions, one for each operation mode coincide. In the middle picture of Fig. 7 r_3 is extended by $3/8$ and the two operation modes provide two different solutions (mode $x_1 = 0$ is tagged with primes, e.g. B_3', and to avoid confusion only one point of the moving systems is tagged). In Fig. 7 (right) the same situation can be observed with r_3 extended by $3/4$.

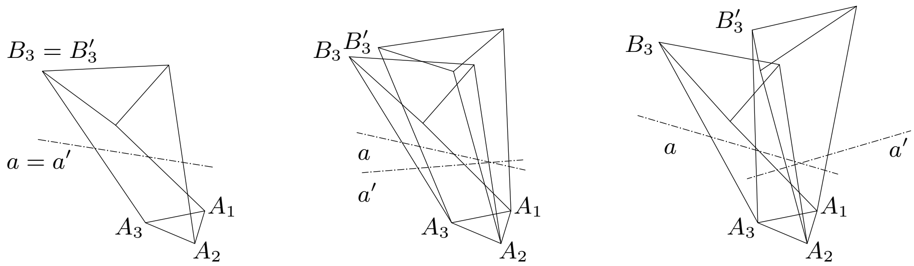

Figure 7: The way out of a singular pose into the two different operation modes

8 Conclusion

The algebraic description of a 3-RPS manipulator with constraint equations turned out to be the key for a complete description of direct kinematics and operation modes. All singularities could be computed in kinematic image space in general and in joint space for any example. It was shown that operation modes can be changed when joint parameters are taken on a 2-dimensional eight degree surface in the joint space. Geometric interpretations of the different operation modes also show that there is a very close connection between the algebraic description and the poses of the manipulator with respect to the screw axis of the transformations.

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