A Reliability Based Robust Multi-objective Optimal Synthesis of Linkage Mechanisms Considering Tolerances

P. Venkateshwara Chowdary, Gurunathan Saravana Kumar, Palaniappan Ramu

Abstract

The paper presents a methodology to synthesis simple planar linkage mechanism involving a multi objective optimization model considering cost associated with mechanical tolerances and the reliability based robustness of generating the intended path. An adapted probabilistic model for the deviation of the actual path generated by a coupler point from the desired one, by considering the structural and mechanical errors due to tolerances and clearances is considered. A synthesis procedure of the linkages is formulated as a multi objective non-linear optimization problem with robustness subject to a reliability level and cost as objectives. The reliability index of the mechanism is based on the probability with which a linkage will generate its intended design motion with specified precision. The robustness is based on the ratio of the mean to standard deviation of the error in the traced path with respect to the target. The cost index is based on manufacturing and assembly cost, which are functions of tolerances and clearances in joints. Multi objective genetic algorithm (MOGA) is employed as the search tool. A Four-bar path generating mechanism is selected for numerical illustration.

Keywords: Mechanism synthesis; tolerances; reliability; cost; multi objective genetic algorithm

1 Introduction

The path generated by a linkage mechanism in general deviates from the desired one. The deviation is due to structural error caused by the link designed dimensions and mechanical error caused by tolerances and clearances in joints. By assigning tight tolerances and clearances, one can synthesis a linkage mechanism whose actual path of the coupler point is quite close to the desired one. But it is very costlier to maintain lower tolerance limits in manufacturing and assembly of linkages. Thus the designer is posed with the challenge to choose appropriate levels of tolerances in a mechanism according to the limitations on manufacturing yet design a mechanism with high reliability. Different methods have been used for the synthesis of linkage mechanisms (Vallejo *et al.* 1995, Haulin *et al.* 2001, Todorov 2002). Attempts have been made to synthesize and analyze mechanical errors in path generating mechanism (Mallik and Dhande 1987). The concept of reliability due to mechanical

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errors caused by tolerances and minimization of the cost (Lee and Woo 1990) has been accomplished for spatial mechanisms (Shi 1997). The relationship between the tolerances to the cost of achieving it in production can be modeled in functional form (He 1991) and the same can be used for optimal allocation (Lee and Woo 1990) and synthesis of linkage mechanism (Shi 1997). Sensitivity analysis of mechanical errors due to tolerances in constituent linkages has been studied (Sharfi and Smith 1983, Pavlović 2007). A robust optimization procedure for path generating mechanism has been presented by considering reliability due to structural and mechanical error (Shi, *et al.* 2005). A robust mechanism synthesis with random and interval variables was developed by employing Monte Carlo simulation for robustness assessment (Due et al., 2009). However, a more generic and a global optimization procedure that considers both robustness and cost associated with the same simultaneously is required so as to design mechanisms with optimal tolerance allocation. Evolutionary methods based on artificial neural networks (ANN), genetic algorithms (GA) and fuzzy logic (FL) have been successfully used for mechanism synthesis (Vasiliu and Yannou 2001, Cabrera *et al.* 2002, Laribi *et al.* 2004). The motivation for the same is that these methods are stochastic in nature and thus are better equipped to find solutions to highly nonlinear and discontinuous problems. Different types of genetic algorithms, because of their high probability in finding the global optimum point and their effectiveness, have been successfully used for optimal synthesis of different mechanisms (Cabrera *et al.* 2002, Laribi *et al.* 2004, Mundo *et al.* 2006). Multi objective optimum synthesis of planar mechanism using genetic algorithms and concept of *pareto* optimality has been presented (Cabrera *et al.* 2007, Nariman-Zadeh *et. al.* 2008). The basic idea of multi-objective optimization is to generate a set of optimal solutions (rather than a single solution) by taking all objectives into account, without a priori assigning greater priority to one objective or the other. Being a population based approach GA's are well suited to solve multi-objective optimization problems. Non-dominated Sorting Genetic Algorithm (NSGA) (Srinivas and Deb 1994) and Fast Non-dominated Sorting Genetic Algorithm (NSGA-II) (Deb et. al. 2002) are well-known and credible algorithms that have been used in many applications and their performances have been tested in several comparative studies.

The main contribution of this paper is a method based on multi objective global optimization to synthesis a planar path generating mechanism by taking cost and robustness as objectives subject to a reliability constraint. The reliability index of the mechanism is based on the probability with which a linkage will generate its intended design motion with specified precision. The cost index is based on manufacturing and assembly cost, which are functions of tolerances and clearances in joints. The robustness is estimated as a ratio of the mean of the error to the standard deviation of the error. The proposed method envisages converging to *pareto* optimal solutions without any *a priori* knowledge or initial solution by using a non-deterministic mathematical optimization technique, Multi Objective Genetic Algorithms (MOGA) (Srinivas and Deb 1994). Section 2 describes the formulation of the multi objective optimization for synthesis of planar mechanism, section 3 provides an illustrative example to present the proposed method and finally section 4 concludes the work.

2 Problem Formulation

The problem considered is path generating planar mechanism synthesis. The objective is to synthesis a robust as well as economical mechanism subject to a reliability level that will trace the path of the coupler point as close as possible to the desired one with optimal allocation of link dimensions and tolerances in manufacturing. The optimal synthesis of linkages provides a set of *pareto* optimal link dimensions for the two conflicting objectives which are the robustness and the cost. The formulation is described subsequently.

2.1 Formulation of reliability constraint

Consider n number of positions that defines the path to be generated by the mechanism with limited deviations in output. Let $\mathbf{q}$ represents s random variables associated with link dimensions and $\Delta\mathbf{q}$ represents the errors in the effective link dimensions due to tolerances and clearances, which are independent random variables with zero means and $s \times s$ diagonal covariance matrix $\mathbf{V}_q = \mathrm{diag}(\sigma_{q_1}^2 \quad \sigma_{q_2}^2 \quad \ldots\ldots \quad \sigma_{q_s}^2)$ (Mallik and Dhande 1987, Shi 1997, Shi et al. 2005). For closed loop linkages like four bar mechanisms set of kinematics loop equations can be written as follows:

$$\mathbf{F}(\mathbf{U}, v, \mathbf{q}) = 0 \tag{1}$$

where $\mathbf{F} = [f_1 \quad f_2]^T$ represents 2 independent loop equations in case of planar mechanisms and $\mathbf{U} = [u_1 \; u_2]^T$ represents 2 outputs of secondary variables, which are functions of the effective link dimensions $\mathbf{q}$ and v is the input of the mechanism. Differentiating the set of Equation (1), one obtains

$$\Delta\mathbf{U} = -\left(\frac{\partial\mathbf{F}}{\partial\mathbf{U}}\right)^{-1} \frac{\partial\mathbf{F}}{\partial\mathbf{q}} \Delta\mathbf{q} \tag{2}$$

where $\frac{\partial\mathbf{F}}{\partial\mathbf{U}}$ is a 2 by 2 Jacobian matrix that is assumed to be reversible. Consider, the coupler point $P(x, y)$, represented in homogeneous coordinate ($\mathbf{r} = [x\, y\; 1]$), in world coordinate frame, ${}^i\mathbf{r}$ (in i^{th} local coordinate frame associated with the i^{th} link) and $\mathbf{T}_i$ the transformations matrices for the i^{th} local coordinate frame to the world coordinate frame, whose components are functions $\mathbf{U}$ and $\Delta\mathbf{q}$. Then,

$$\mathbf{r} = \mathbf{T}_i \; {}^i\mathbf{r} \tag{3}$$

Equation (3) can be expanded using Taylor series (neglecting second and higher order terms) about the mean values of the individual random variables and substituting, Equation (2) one gets

$$\begin{aligned} \mathbf{r} &= \overline{\mathbf{T}}_i \; {}^i\overline{\mathrm{r}} + \sqrt{\left[\mathbf{W}_2 - \mathbf{W}_1 \left(\frac{\partial\mathbf{F}}{\partial\mathbf{U}}\right)^{-1} \frac{\partial\mathbf{F}}{\partial\mathbf{q}}\right] \mathbf{V}_q \left[\mathbf{W}_2 - \mathbf{W}_1 \left(\frac{\partial\mathbf{F}}{\partial\mathbf{U}}\right)^{-1} \frac{\partial\mathbf{F}}{\partial\mathbf{q}}\right]} \\ &= \overline{\mathbf{r}} + \sqrt{\mathbf{V}_r} \end{aligned} \tag{4}$$

where $\mathbf{W}_1$, $\mathbf{W}_2$ are 3×2 and 3×2 matrix respectively. The vector $\mathbf{r}$ that is of linear form given in Equation (4) is a normal vector with mean $\overline{\mathbf{r}}$ and 3×3 covariance matrix $\mathbf{V}_r$ (Shi et al. 2005).

The probability with which the given mechanism with structural and mechanical errors will reach a given desired position with specified precision is said to be reliability and is formulated as a positive reliability index β (Shi 1997, Shi et al. 2005). Assuming that the actual position of a coupler point of the planar mechanism $P = [x, y]$ is required to be quite close to the desired one $P_d = [x_d, y_d]$. The deviations between them, denoted by **Z**, have the following statistical characteristics:

$$\mathbf{Z} = [x - x_d, y - y_d]^T \text{ with } \overline{\mathbf{Z}} = [\overline{x} - x_d, \overline{y} - y_d]^T \text{ and } \mathbf{V}_z = \mathbf{V}_r^2 \tag{5}$$

where $\overline{\mathbf{Z}}$ and $\mathbf{V}_z$ represents the structural errors and the mechanical errors respectively. $\mathbf{V}_r^{\,2}$ is the upper left 2×2 matrix of the covariance matrix $\mathbf{V}_r$ in Equation (5). The probability with which a linkage will generate its intended design motion with specified precision (reliability) can be computed by the integral of a normal probability density function $\phi(\overline{\mathbf{Z}}, \mathbf{V}_z)$ for the coupler point for a specified reliable region bounded by design functions $g_i(\mathbf{Z}) \geq 0$, i = 1,2 that are limit-state surface corresponding to the deviations in x and y (for planar mechanisms). A reasonable approximation to this probability can be bounded by β_i^j, known as the reliability index decided by i^{th} design function $g_i(\mathbf{Z})$ for the j^{th} design position, which is the minimum distance from the origin to i^{th} limit-state surface in an independent standardized coordinate system and can be evaluated as:

$$\beta_i^j = \min_{q \in g_i\left(\overline{\mathbf{Z}}\right) = 0} \left\{ \left[\overline{\mathbf{Z}}\text{-}\overline{\mathbf{Z}}^{(j)} \right]^{\mathrm{T}} V_q^{-1} \left[\overline{\mathbf{Z}}\text{-}\overline{\mathbf{Z}}^{(j)} \right] \right\}^{\frac{1}{2}} \tag{6}$$

The design solution $\mathbf{Z}^*$ satisfying Equation (6) can be searched by means of using an iterative scheme (Shi et al. 2005). In the present work, the following reliability function is adapted as one of the constraints for synthesis:

$$\beta(\mathbf{q}^*) = \min_{k=1}^{n} \{\beta(\mathbf{q}^*, \mathbf{V}_q, v^{(q)})\} \tag{7}$$

where β_i^j is the reliability index as per Equation (6), v is an input or primary variable, $\mathbf{q}^*$ is mean link lengths and $\mathbf{V}_q$ involves tolerances and clearances. The constraints are modelled from restrictions in geometry and performance as,

$$h_i(\mathbf{q}^*) - \xi \left[\frac{\partial h_i}{\partial \mathbf{q}} \mathbf{V}_q \left(\frac{\partial h_i}{\partial \mathbf{q}} \right)^T \right]^{\frac{1}{2}} \geq 0 \tag{8}$$

where ξ is the confidence level which can be derived strictly from the normal distribution according to the confidence level which can be generally given by designers. In practice three sigma confidence levels is considered and thus ξ can be assigned 3 unless otherwise the designer wants stringent confidence.

2.2 Formulation of cost objective

The tolerances to be achieved in manufacturing linkages and the constraints in assembly of these linkages with tight clearances imply a higher cost of realizing a mechanism designed with tight tolerances and clearance limits. From a survey (He

1991), of the published literature, the relationship between the component tolerance *t* to the cost of achieving it in production *C* (*t*) can be modeled in functional form. In the present work, the cost is formulated as:

$$C(\mathbf{q}^*) = \sum_{k=1}^{s} \frac{1}{t_i} \tag{9}$$

where t_i are the tolerances corresponding to *s* link dimensions.

2.3 Formulation of the robustness objective

Different formulation techniques are available for framing a robustness objective (Park et al., 2006). Here, the robustness formulation follows the standard Taguchi's signal to noise concept (Phadke, 1989). Robustness index is obtained as:

$$\eta = 10\log_{10}\left(\frac{\mu^2}{\sigma^2}\right) \tag{10}$$

where η is the robustness index, μ the mean of the error and, σ the standard deviation of the error. Error here refers to the error between the patch traced by the mechanism to the desired.

3 Numerical Example – Four Bar Mechanism

Consider a planar four-bar mechanism for path generation as shown schematically in Figure 1. This example is adapted from Shi *et al.* 2005 to demonstrate the proposed method. It is required that the actual path of the coupler point *P* is as close as possible to the desired points $P_d^{(k)} = \left\{x_d^{(k)},\ y_d^{(k)}\right\}$, *k*=1, 2, . . . *n* = 9 (Table 1). The design variables are link dimensions, input link position, global translation and rotation angle, coupler link angle and the tolerances, $[l_1, l_2, l_3, l_4, l_5, \varphi_1^{(0)}, x_a, y_a, \alpha, \beta, t_1, t_2, t_3, t_4]$. The tolerances t_i are specified as percentages x_i of link dimensions l_i and radial clearances for every joint is taken as 0.5mm. Let the given permitted output deviation limits be $\varepsilon_{x\max}$ = 2mm and ε_{ymax} = 3mm.

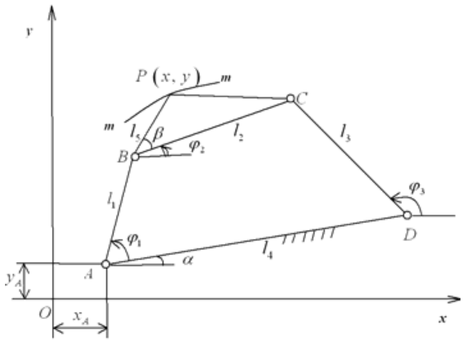

Figure 1. Planar Four Bar Mechanism

Table 1: Desired position of coupler points

k	$\varphi_1^{(0k)}\ (^\circ)$	$x_d^{(k)}$	$y_d^{(k)}$
1	0	24	65
2	40	14.5	69
3	80	0	62.5
4	120	-10.5	50.5
5	160	-13	36
6	200	-7.5	26.5
7	240	3	27
8	280	14.5	37.5
9	320	23	52.5

The multi objective optimization is formulated as:

$$\textit{Minimize:}\ \ f_1 = C(\mathbf{q}^*) = \sum_{i=1}^{4} \frac{1}{t_i} \tag{11}$$

$$\textit{Maximize:}\ f_2 = \eta = 10\log_{10}\left(\frac{{\mu_{er}}^2}{{\sigma_{er}}^2}\right) \tag{12}$$

where $t_i = x_i \times l_i$ are tolerances corresponding to i =1,2..4 links. The reliability index $\beta(\mathbf{q}^*)$ corresponds to the worst case of 2 design functions (corresponding to deviations in x and y) to the n = 9 design position. The constraints are,

Subjected to

$$1)\quad h_i(\mathbf{q}^*) - 3\left[\frac{\partial h_i}{\partial \mathbf{q}} \mathbf{V}_q \left(\frac{\partial h_i}{\partial \mathbf{q}}\right)\right]^{\frac{1}{2}} \geq 0, \quad i = 1, 2, \ldots, 8$$

$$\left.\begin{aligned} h_1 &= -l_1 - l_2 + l_3 + l_4 \\ h_2 &= -l_1 + l_2 - l_3 + l_4 \\ h_3 &= -l_1 + l_2 + l_3 - l_4 \end{aligned}\right\} \text{ Crank existence conditions} \tag{13}$$

$h_i = l_{i-3}$, $i = 4,5,6,7,8$ are nonnegative conditions of the link dimensions.

$$2)\ \beta(\mathbf{q}^*) = \min_{k=1}^{n}\left\{\min\left[\frac{\varepsilon_{x\max} - \left|\overline{x}^{(k)} - x_d^{(k)}\right|}{\sigma_x}, \frac{\varepsilon_{y\max} - \left|\overline{y}^{(k)} - y_d^{(k)}\right|}{\sigma_y}\right]\right\} \geq 3 \tag{14}$$

The optimization problem thus formulated is solved using MOGA. The following GA parameters were chosen; population size = 100, maximum no. of generations =500, cross-over probability = 0.9, mutation probability = 0.1 and distribution index for crossover and mutation = 20. Convergence (in 200 MOGA generation) leads to the final set of non-dominating solutions, which represent the *pareto* optimal front as shown in Figure 2. It is also clear from Figure 2 that choosing a better value for one objective function in these *pareto* optimal front would cause a worse value of another objective function. There are some significant optimal design facts for the chosen objective functions that can be observed in the *pareto* optimal front. In Figure 2 point B is a significant optimal design point. The mechanism solutions between point A and B have almost same robustness but with increasing cost from point A to B. Such important design facts could not have been found without obtaining the *pareto* front which is made feasible by the use of multi-objective optimization approach for the mechanism synthesis. It should be noted that all the points in the *pareto* have a reliability index of more than 3. However Shi et al (2005) obtained a design solution where the reliability index was 1.44. This was obtained by maximizing the reliability index. The path traced by the coupler points are shown in Figure 3 along with the upper and lower bounds in error.

4 Conclusion

The paper proposes an approach to generate *pareto* optimal solutions to mechanism synthesis problem so that the designer can choose from the set of solutions under multiple contradicting objectives. A global optimization method based on multi-objective genetic algorithm was used to optimally design path generating mechanisms taking into account both robustness subject to reliability and cost associated with mechanical errors caused by tolerances and clearances. The multi-objective optimum synthesis of mechanisms led to discovering of some significant trade-offs among these objective functions. Many constraints in real world problems like cost and /or robustness can not be *a priorily* fixed or taken strictly as constraints, since these functions can be shifted in the objective function space so as to allow for greater feasible domain for searching optimal solutions. The multi-objective optimum synthesis of mechanisms procedure illustrates this fact where either the cost / reliability was modeled as objective and reliability/ cost was modeled as constraints (as the case may be) in the reported literature could be modeled as conflicting objectives and *pareto* optimal solutions could be generated for the designer to choose *posteriorly*.

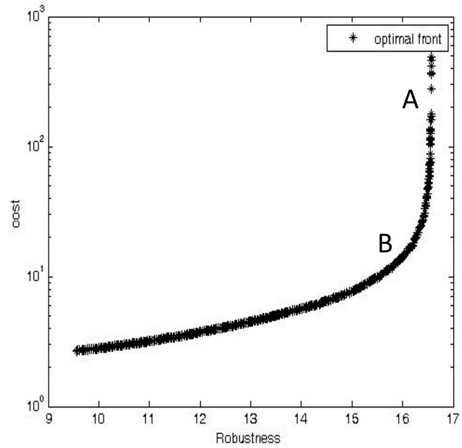

Figure 2. Pareto optimal front of the robustness and cost

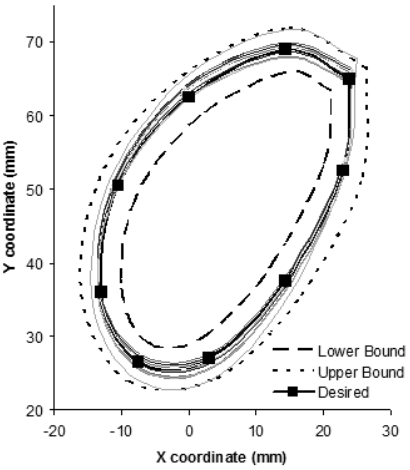

Figure 3. Path generated by the four-bar mechanism synthesized by best reliable and best cost solution obtained by MOGA

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