Kinematic Analysis of a 3-UPU Parallel Manipulator Using Exponential Rotation Matrices

Gökhan Kiper, Eres Söylemez

Abstract

This study addresses the forward kinematics problem of a new type of 3-UPU parallel manipulator. Orientation and position vector loop equations are derived using exponential rotation matrices. It is found that for generic dimensions the manipulator has 4 (Schönflies type) or 3 degrees-of-freedom (dof) depending on whether the platform is assembled parallel to the base or not. There are several assembly modes in both of the cases. When the base and the platform are identical the manipulator is overconstrained with 4 dof. This study also compares vector and quaternion approaches via analysis of the 3-UPU manipulator.

Keywords: Parallel manipulator, Forward kinematics, Exponential rotation matrices

1 Introduction

A new type of 3-universal-prismatic-universal (UPU) parallel manipulator (PM) is presented in [1]. The forward kinematics of this PM is performed using Study parametrization of motion (See ex. [2]). The analysis showed that for generic dimensions the PM is kinematotropic with 4 or 3 degrees-of-freedom (dof) depending on whether the platform is assembled parallel to the base or not. When the base and the platform are identical, the PM is overconstrained with 4 dof ([1]).

In this study the forward kinematics of the PM is analysed using the exponential rotation matrices, following [3, 4]. With this analysis tool, more light is shed on the special geometry and motion of the PM.

2 Brief Review of Özgören's notation

In [4], Özgören introduces a convention for the kinematic description of mechanisms as an alternative to the well known Denavit-Hartenberg convention [5]. With this convention for example the location of frame origin O_b of link b (joined to link a via joint ab) with respect to frame origin O_a of link a expressed in frame of link a is

$$\mathbf{r}_{a,b}^{(a)} = \mathbf{r}_{a,ab}^{(a)} + \mathbf{C}^{(a,ab)}\mathbf{r}_{ab,ba}^{(ab)} + \mathbf{C}^{(a,b)}\mathbf{r}_{ba,b}^{(b)}$$

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where, the rotation matrices $\mathbf{C}^{(a,ab)}$, etc are expressed in terms of exponential rotation matrices $e^{\tilde{\mathbf{n}}\theta}$, $\mathbf{n}$ representing the axis of rotation and θ being the amount of rotation.

3 Description of the Manipulator

The universal joints of the PM are arranged such that the revolution axes associated with the base or platform are all parallel to each other and also for each limb the associated two revolution axes are parallel to each other. To illustrate via Fig. (1), R_{11} // R_{21} // R_{31} for the base, R_{14} // R_{24} // R_{34} for the platform and R_{i2} // R_{i3} for each limb i = 1, 2, 3. Also we limit our focus on the case where R_{i2} and R_{i3} are coplanar for the base and the platform, respectively.

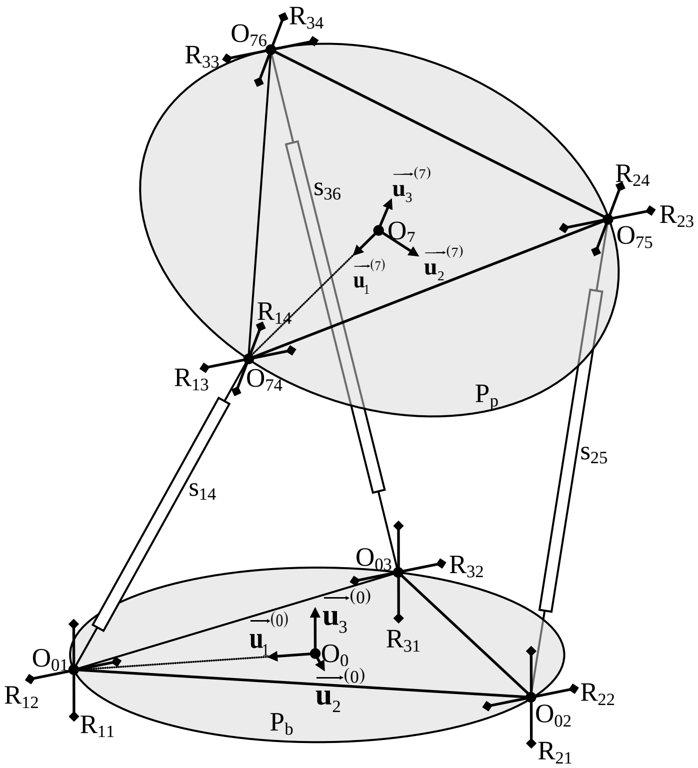

Figure 1: The 3-UPU PM. It has three limbs and all limbs connect the platform to the base via universal-prismatic-universal joint sequence.

Since R_{i2} and R_{i3} reside in their respective planes – the base plane P_b and the platform plane P_p – the revolution axes intersections O_{01}, O_{02} and O_{03} of the base and O_{74}, O_{75} and O_{76} of the platform define triangles in P_b and P_p, respectively. Locate a fixed and a moving coordinate frame at the circumcenter of these triangles – $O_0\left(\vec{\mathbf{u}}_1^{(0)}, \vec{\mathbf{u}}_2^{(0)}, \vec{\mathbf{u}}_3^{(0)}\right)$ and $O_7\left(\vec{\mathbf{u}}_1^{(7)}, \vec{\mathbf{u}}_2^{(7)}, \vec{\mathbf{u}}_3^{(7)}\right)$ such that $\vec{\mathbf{u}}_1^{(0)} / \vec{\mathbf{u}}_1^{(7)}$ axis points towards O_{01}/O_{74} and $\vec{\mathbf{u}}_1^{(0)}\,\vec{\mathbf{u}}_2^{(0)} / \vec{\mathbf{u}}_1^{(7)}\,\vec{\mathbf{u}}_2^{(7)}$ -plane is coincident with P_b/P_p. Let the coordinates of O_{01}, O_{02}, O_{03} and of O_{74}, O_{75}, O_{76} with respect to (wrt) the specified frames be

$$O_{01}(R,0,0), O_{02}(R\cos\alpha_{12}, R\sin\alpha_{12}, 0), O_{31}(R\cos\alpha_{13}, R\sin\alpha_{13}, 0) \text{ wrt } O_0\left(\vec{\mathbf{u}}_1^{(0)}, \vec{\mathbf{u}}_2^{(0)}, \vec{\mathbf{u}}_3^{(0)}\right)$$
$$O_{74}(r,0,0), O_{75}(r\cos\alpha_{45}, r\sin\alpha_{45}, 0), O_{76}(r\cos\alpha_{46}, r\sin\alpha_{46}, 0) \text{ wrt } O_7\left(\vec{\mathbf{u}}_1^{(7)}, \vec{\mathbf{u}}_2^{(7)}, \vec{\mathbf{u}}_3^{(7)}\right)$$

4 Forward Kinematics for the Generic Case

In [4], the orientation equations are derived as.

$$\mathbf{C}^{(0,7)} = \mathbf{C}^{(0,1)}\mathbf{C}^{(1,4)}\mathbf{C}^{(4,7)} = \mathbf{C}^{(0,01)}\mathbf{C}^{(01,10)}\mathbf{C}^{(10,1)}\mathbf{C}^{(1,14)}\mathbf{C}^{(14,41)}\mathbf{C}^{(41,4)}\mathbf{C}^{(4,47)}\mathbf{C}^{(47,74)}\mathbf{C}^{(74,7)} \quad (1)$$
$$\mathbf{C}^{(0,7)} = \mathbf{C}^{(0,2)}\mathbf{C}^{(2,5)}\mathbf{C}^{(5,7)} = \mathbf{C}^{(0,02)}\mathbf{C}^{(02,20)}\mathbf{C}^{(20,2)}\mathbf{C}^{(2,25)}\mathbf{C}^{(25,52)}\mathbf{C}^{(52,5)}\mathbf{C}^{(5,57)}\mathbf{C}^{(57,75)}\mathbf{C}^{(75,7)} \quad (2)$$
$$\mathbf{C}^{(0,7)} = \mathbf{C}^{(0,3)}\mathbf{C}^{(3,6)}\mathbf{C}^{(6,7)} = \mathbf{C}^{(0,03)}\mathbf{C}^{(03,30)}\mathbf{C}^{(30,3)}\mathbf{C}^{(3,36)}\mathbf{C}^{(36,63)}\mathbf{C}^{(63,6)}\mathbf{C}^{(6,67)}\mathbf{C}^{(67,76)}\mathbf{C}^{(76,7)} \quad (3)$$

For the universal joints attached to the base

$$\mathbf{C}^{(01,10)} = e^{\tilde{\mathbf{u}}_3\theta_{01}}e^{\tilde{\mathbf{u}}_1\phi_{01}},\ \mathbf{C}^{(02,20)} = e^{\tilde{\mathbf{u}}_3\theta_{02}}e^{\tilde{\mathbf{u}}_1\phi_{02}},\ \mathbf{C}^{(03,30)} = e^{\tilde{\mathbf{u}}_3\theta_{03}}e^{\tilde{\mathbf{u}}_1\phi_{03}} \quad (4)$$

For the prismatic joints

$$\mathbf{C}^{(14,41)} = \mathbf{C}^{(25,52)} = \mathbf{C}^{(36,63)} = \mathbf{I} \quad (5)$$

For the universal joints attached to the platform

$$\mathbf{C}^{(47,74)} = e^{\tilde{\mathbf{u}}_1\phi_{47}}e^{\tilde{\mathbf{u}}_3\theta_{47}},\ \mathbf{C}^{(57,75)} = e^{\tilde{\mathbf{u}}_1\phi_{57}}e^{\tilde{\mathbf{u}}_3\theta_{57}},\ \mathbf{C}^{(67,76)} = e^{\tilde{\mathbf{u}}_1\phi_{67}}e^{\tilde{\mathbf{u}}_3\theta_{67}} \quad (6)$$

Note that all θs are associated with base/platform joints and all ϕs are associated with the limbs. For the constant rotation matrices

$$\mathbf{C}^{(0,01)} = \mathbf{I},\ \mathbf{C}^{(0,02)} = e^{\tilde{\mathbf{u}}_3\alpha_{12}},\ \mathbf{C}^{(0,03)} = e^{\tilde{\mathbf{u}}_3\alpha_{13}} \quad (7)$$
$$\mathbf{C}^{(7,74)} = \mathbf{I},\ \mathbf{C}^{(7,75)} = e^{\tilde{\mathbf{u}}_3\alpha_{45}},\ \mathbf{C}^{(7,76)} = e^{\tilde{\mathbf{u}}_3\alpha_{46}} \quad (8)$$

Take the prismatic joint frames coincident with the relevant revolute joint frames:

$$\begin{aligned}\mathbf{C}^{(1,10)} = \mathbf{C}^{(2,20)} = \mathbf{C}^{(3,30)} = \mathbf{C}^{(1,14)} = \mathbf{C}^{(2,25)} = \mathbf{C}^{(3,36)} = \mathbf{I}\\ \mathbf{C}^{(4,47)} = \mathbf{C}^{(5,57)} = \mathbf{C}^{(6,67)} = \mathbf{C}^{(4,41)} = \mathbf{C}^{(5,52)} = \mathbf{C}^{(6,63)} = \mathbf{I}\end{aligned} \quad (9)$$

Then Eqs. (1-3) become

$$\mathbf{C}^{(0,7)} = e^{\tilde{\mathbf{u}}_3\theta_{01}}e^{\tilde{\mathbf{u}}_1(\phi_{01}+\phi_{47})}e^{\tilde{\mathbf{u}}_3\theta_{47}} \quad (10)$$
$$\mathbf{C}^{(0,7)} = e^{\tilde{\mathbf{u}}_3(\alpha_{12}+\theta_{02})}e^{\tilde{\mathbf{u}}_1(\phi_{02}+\phi_{57})}e^{\tilde{\mathbf{u}}_3(\theta_{57}-\alpha_{45})} \quad (11)$$
$$\mathbf{C}^{(0,7)} = e^{\tilde{\mathbf{u}}_3(\alpha_{13}+\theta_{03})}e^{\tilde{\mathbf{u}}_1(\phi_{03}+\phi_{67})}e^{\tilde{\mathbf{u}}_3(\theta_{67}-\alpha_{46})} \quad (12)$$

From Eqs. (10-12)

$$e^{\tilde{\mathbf{u}}_3(\alpha_{12}+\theta_{02})} e^{\tilde{\mathbf{u}}_1(\phi_{02}+\phi_{57})} e^{\tilde{\mathbf{u}}_3(\theta_{57}-\alpha_{45})} = e^{\tilde{\mathbf{u}}_3\theta_{01}} e^{\tilde{\mathbf{u}}_1(\phi_{01}+\phi_{47})} e^{\tilde{\mathbf{u}}_3\theta_{47}} \tag{13}$$

$$e^{\tilde{\mathbf{u}}_3(\alpha_{13}+\theta_{03})} e^{\tilde{\mathbf{u}}_1(\phi_{03}+\phi_{67})} e^{\tilde{\mathbf{u}}_3(\theta_{67}-\alpha_{46})} = e^{\tilde{\mathbf{u}}_3\theta_{01}} e^{\tilde{\mathbf{u}}_1(\phi_{01}+\phi_{47})} e^{\tilde{\mathbf{u}}_3\theta_{47}} \tag{14}$$

In [4], the location equations are derived for the three limbs as

$$\mathbf{r}_{0,7}^{(0)} = R\mathbf{u}_1 + e^{\tilde{\mathbf{u}}_3\theta_{01}} e^{\tilde{\mathbf{u}}_1\phi_{01}} \left(s_{14}\mathbf{u}_3 - r e^{\tilde{\mathbf{u}}_1\phi_{47}} e^{\tilde{\mathbf{u}}_3\theta_{47}} \mathbf{u}_1\right) \tag{15}$$

$$\mathbf{r}_{0,7}^{(0)} = e^{\tilde{\mathbf{u}}_3\alpha_{12}} \left[R\mathbf{u}_1 + e^{\tilde{\mathbf{u}}_3\theta_{02}} e^{\tilde{\mathbf{u}}_1\phi_{02}} \left(s_{25}\mathbf{u}_3 - r e^{\tilde{\mathbf{u}}_1\phi_{57}} e^{\tilde{\mathbf{u}}_3\theta_{57}} \mathbf{u}_1\right)\right] \tag{16}$$

$$\mathbf{r}_{0,7}^{(0)} = e^{\tilde{\mathbf{u}}_3\alpha_{13}} \left[R\mathbf{u}_1 + e^{\tilde{\mathbf{u}}_3\theta_{03}} e^{\tilde{\mathbf{u}}_1\phi_{03}} \left(s_{36}\mathbf{u}_3 - r e^{\tilde{\mathbf{u}}_1\phi_{67}} e^{\tilde{\mathbf{u}}_3\theta_{67}} \mathbf{u}_1\right)\right] \tag{17}$$

From Eqs. (15-17)

$$\begin{aligned} & e^{\tilde{\mathbf{u}}_3\alpha_{12}} \left[R\mathbf{u}_1 + e^{\tilde{\mathbf{u}}_3\theta_{02}} e^{\tilde{\mathbf{u}}_1\phi_{02}} \left(s_{25}\mathbf{u}_3 - r e^{\tilde{\mathbf{u}}_1\phi_{57}} e^{\tilde{\mathbf{u}}_3\theta_{57}} \mathbf{u}_1\right)\right] \\ & = R\mathbf{u}_1 + e^{\tilde{\mathbf{u}}_3\theta_{01}} e^{\tilde{\mathbf{u}}_1\phi_{01}} \left(s_{14}\mathbf{u}_3 - r e^{\tilde{\mathbf{u}}_1\phi_{47}} e^{\tilde{\mathbf{u}}_3\theta_{47}} \mathbf{u}_1\right) \end{aligned} \tag{18}$$

$$\begin{aligned} & e^{\tilde{\mathbf{u}}_3\alpha_{13}} \left[R\mathbf{u}_1 + e^{\tilde{\mathbf{u}}_3\theta_{03}} e^{\tilde{\mathbf{u}}_1\phi_{03}} \left(s_{36}\mathbf{u}_3 - r e^{\tilde{\mathbf{u}}_1\phi_{67}} e^{\tilde{\mathbf{u}}_3\theta_{67}} \mathbf{u}_1\right)\right] \\ & = R\mathbf{u}_1 + e^{\tilde{\mathbf{u}}_3\theta_{01}} e^{\tilde{\mathbf{u}}_1\phi_{01}} \left(s_{14}\mathbf{u}_3 - r e^{\tilde{\mathbf{u}}_1\phi_{47}} e^{\tilde{\mathbf{u}}_3\theta_{47}} \mathbf{u}_1\right) \end{aligned} \tag{19}$$

With the prismatic inputs s_{14}, s_{25} and s_{36}, Eqs. (13-14, 18-19) are to be solved for angles θ_{01}, ϕ_{01}, ϕ_{47}, θ_{47}, θ_{02}, ϕ_{02}, ϕ_{57}, θ_{57}, θ_{03}, ϕ_{03}, ϕ_{67} and θ_{67}. the limbs. From Eq. (13)

$$e^{\tilde{\mathbf{u}}_1(\phi_{02}+\phi_{57})} e^{\tilde{\mathbf{u}}_3(\theta_{57}-\alpha_{45}-\theta_{47})} = e^{\tilde{\mathbf{u}}_3(\theta_{01}-\alpha_{12}-\theta_{02})} e^{\tilde{\mathbf{u}}_1(\phi_{01}+\phi_{47})} \tag{20}$$

Pre and post multiplying Eq. (20) by $\mathbf{u}_1^T$ and $\mathbf{u}_3$ we get

$$\sin(\theta_{01}-\alpha_{12}-\theta_{02})\sin(\phi_{01}+\phi_{47}) = 0 \tag{21}$$

Pre and post multiplying Eq. (20) by $\mathbf{u}_3^T$ and $\mathbf{u}_1$ we get

$$\sin(\theta_{57}-\alpha_{45}-\theta_{47})\sin(\phi_{02}+\phi_{57}) = 0 \tag{22}$$

Similarly from Eq. (14)

$$\sin(\theta_{01}-\alpha_{13}-\theta_{03})\sin(\phi_{01}+\phi_{47}) = 0 \tag{23}$$

$$\sin(\theta_{67}-\alpha_{46}-\theta_{47})\sin(\phi_{03}+\phi_{67}) = 0 \tag{24}$$

From Fig. (1) and Eqs. (21-24) notice that the following case occurs when P_b // P_p:

$$\phi_{01} + \phi_{47} = \phi_{02} + \phi_{57} = \phi_{03} + \phi_{67} = 0 \text{ or } \pi \quad (25)$$

We shall call this mode as the parallel mode (See [1]). For the nonparallel mode

$$\theta_{01} - \alpha_{12} - \theta_{02} = \theta_{57} - \alpha_{45} - \theta_{47} = \theta_{01} - \alpha_{13} - \theta_{03} = \theta_{67} - \alpha_{46} - \theta_{47} = 0 \text{ or } \pi \quad (26)$$

Eq. (26) implies that in the non-parallel mode the revolute joint axes associated with the base/platform are parallel, i.e. R_{12} // R_{22} // R_{32} and R_{13} // R_{23} // R_{33} (See Fig. (1)).

Leaving rigorousness aside we shall assume that either Eq. (25) or Eq. (26) is the case, but not any other combination. We shall continue our analysis for the parallel and non-parallel modes separately.

4.1 Non-Parallel Mode

From Eq. (26)

$$\theta_{02} = \theta_{01} - \alpha_{12},\ \theta_{03} = \theta_{01} - \alpha_{13}, \theta_{57} = \theta_{47} + \alpha_{45}, \theta_{67} = \theta_{47} + \alpha_{46} \quad (27)$$

Without loss of generality $+\pi$ possibility may be disregarded, because that case just corresponds to a limb flipped along its axis, which has no effect on the platform pose. Then using Eq. (27) from Eqs. (13-14)

$$\phi_{01} + \phi_{47} = \phi_{02} + \phi_{57} = \phi_{03} + \phi_{67} \quad (28)$$

Together with Eq. (27), Eq. (28) means that the angle between P_b and P_p is $\phi_{01} + \phi_{47}$ and R_{12} // R_{22} // R_{32} // R_{13} // R_{23} // R_{33} are parallel to the line of intersection of P_b and P_p. These conditions were stated via geometric reasoning and was implemented as constraints in [1], however it is revealed via equations here.

Implementing Eqs. (27-28) in Eqs. (18-19) we get

$$\left[\mathrm{R} e^{-\tilde{\mathbf{u}}_3\theta_{01}} \left(\mathbf{I} - e^{\tilde{\mathbf{u}}_3\alpha_{12}} \right) - \mathrm{r} e^{\tilde{\mathbf{u}}_1(\phi_{01}+\phi_{47})} e^{\tilde{\mathbf{u}}_3\theta_{47}} \left(\mathbf{I} - e^{\tilde{\mathbf{u}}_3\alpha_{45}} \right) \right] \mathbf{u}_1 = \left(s_{25} e^{\tilde{\mathbf{u}}_1\phi_{02}} - s_{14} e^{\tilde{\mathbf{u}}_1\phi_{01}} \right) \mathbf{u}_3 \quad (29)$$

$$\left[\mathrm{R} e^{-\tilde{\mathbf{u}}_3\theta_{01}} \left(\mathbf{I} - e^{\tilde{\mathbf{u}}_3\alpha_{13}} \right) - \mathrm{r} e^{\tilde{\mathbf{u}}_1(\phi_{01}+\phi_{47})} e^{\tilde{\mathbf{u}}_3\theta_{47}} \left(\mathbf{I} - e^{\tilde{\mathbf{u}}_3\alpha_{46}} \right) \right] \mathbf{u}_1 = \left(s_{36} e^{\tilde{\mathbf{u}}_1\phi_{03}} - s_{14} e^{\tilde{\mathbf{u}}_1\phi_{01}} \right) \mathbf{u}_3 \quad (30)$$

The scalar equations corresponding to (29-30) are:

$$\left[c\theta_{01} - c(\theta_{01} - \alpha_{12}) \right] R - \left[c\theta_{47} - c(\theta_{47} + \alpha_{45}) \right] r = 0 \quad (31)$$

$$\left[s\theta_{01} - s(\theta_{01} - \alpha_{12}) \right] R + c(\phi_{01} + \phi_{47}) \left[s\theta_{47} - s(\theta_{47} + \alpha_{45}) \right] r + s\phi_{01} s_{14} = s\phi_{02} s_{25} \quad (32)$$

$$-s(\phi_{01} + \phi_{47}) \left[s\theta_{47} - s(\theta_{47} + \alpha_{45}) \right] r + c\phi_{01} s_{14} = c\phi_{02} s_{25} \quad (33)$$

$$\left[c\theta_{01} - c(\theta_{01} - \alpha_{13}) \right] R - \left[c\theta_{47} - c(\theta_{47} + \alpha_{46}) \right] r = 0 \quad (34)$$

$$\left[s\theta_{01} - s(\theta_{01} - \alpha_{13}) \right] R + c(\phi_{01} + \phi_{47}) \left[s\theta_{47} - s(\theta_{47} + \alpha_{46}) \right] r + s\phi_{01} s_{14} = s\phi_{03} s_{36} \quad (35)$$

$$-s\left(\phi_{01}+\phi_{47}\right)\left[s\theta_{47}-s\left(\theta_{47}+\alpha_{46}\right)\right]r+c\phi_{01}s_{14}=c\phi_{03}s_{36} \tag{36}$$

Analytic solution seems not possible for these equations. We proceed with a numerical solution algorithm:

1. Assume θ_{01} is known and then solve for θ_{47} from Eq. (31) using either tangent of half angle substitution or phase angle method (See ex. [6]), hence obtain two θ_{47}'s for given θ_{01}.
2. Substitute $\theta_{47}(\theta_{01})$ from Step 1 in Eq. (34) and numerically solve for θ_{01}.
3. Eliminate ϕ_{02} from Eqs. (32-33) and ϕ_{03} from Eqs. (35-36) by leaving $s\phi_{02}$ (or $s\phi_{03}$) and $c\phi_{02}$ (or $c\phi_{03}$) alone, square up and add up to get 1. The resulting equations are:

$$\begin{aligned}&2\left[s\theta_{47}-s\left(\theta_{47}+\alpha_{45}\right)\right]\left\{s\phi_{01}s_{14}+\left[s\theta_{01}-s\left(\theta_{01}-\alpha_{12}\right)\right]R\right\}rc\left(\phi_{01}+\phi_{47}\right)\\&-2\left[s\theta_{47}-s\left(\theta_{47}+\alpha_{45}\right)\right]c\phi_{01}rs_{14}s\left(\phi_{01}+\phi_{47}\right)+2\left[s\theta_{01}-s\left(\theta_{01}-\alpha_{12}\right)\right]s\phi_{01}Rs_{14}\\&+\left[s\theta_{01}-s\left(\theta_{01}-\alpha_{12}\right)\right]^2R^2+\left[s\theta_{47}-s\left(\theta_{47}+\alpha_{45}\right)\right]^2r^2+s_{14}{}^2-s_{25}{}^2=0\end{aligned} \tag{37}$$

$$\begin{aligned}&2\left[s\theta_{47}-s\left(\theta_{47}+\alpha_{46}\right)\right]\left\{s\phi_{01}s_{14}+\left[s\theta_{01}-s\left(\theta_{01}-\alpha_{13}\right)\right]R\right\}rc\left(\phi_{01}+\phi_{47}\right)\\&-2\left[s\theta_{47}-s\left(\theta_{47}+\alpha_{46}\right)\right]c\phi_{01}rs_{14}s\left(\phi_{01}+\phi_{47}\right)+2\left[s\theta_{01}-s\left(\theta_{01}-\alpha_{13}\right)\right]s\phi_{01}Rs_{14}\\&+\left[s\theta_{01}-s\left(\theta_{01}-\alpha_{13}\right)\right]^2R^2+\left[s\theta_{47}-s\left(\theta_{47}+\alpha_{46}\right)\right]^2r^2+s_{14}{}^2-s_{36}{}^2=0\end{aligned} \tag{38}$$

   Assume ϕ_{01} is known and solve for $\phi_{01}+\phi_{47}$ from Eq. (37) similar to Step 1 and obtain two values of $\phi_{01}+\phi_{47}$ for given ϕ_{01}.

4. Substitute $(\phi_{01}+\phi_{47})(\phi_{01})$ obtained in Step 3 in Eq. (38) and solve for ϕ_{01}.
5. Leave $s\phi_{02}$ (or $s\phi_{03}$) and $c\phi_{02}$ ($c\phi_{03}$) alone in Eqs. (32-33) (Eqs. (35-36)) and uniquely solve for ϕ_{02} (ϕ_{03}) using atan2 for each set of (θ_{01}, θ_{47}, ϕ_{01}, ϕ_{47}).

Note that tangent of half angle substitution for Eqs. (31) and (34) results in a total algebraic order of 16, hence we expect at most 16 pairs of real solutions for θ_{01} and θ_{47}. For ϕ_{01} and ϕ_{47} the order is 9, so totally we expect at most 144 real solutions. This number is far larger than 8 - the number of solutions obtained in [1] via simulations for the generic case.

It is interesting to see that Eqs. (31) and (34) do not involve the prismatic inputs at all. This means that the angles θ_{01} and θ_{47}, and hence by Eq. (27) θ_{02}, θ_{03}, θ_{57}, and θ_{67} do not depend on the prismatic inputs, but are determined by base and platform geometry only.

To be able to compare the results, the example in [1] with R = 100, r = 80, α_{12} = 130°, α_{13} = 220°, α_{45} = 100°, α_{46} = 260°, s_{14} = 190, s_{25} = 200, s_{36} = 210 units is re-solved using (27-28) and the algorithm above. The computations are performed using Microsoft Excel® and the Goal-Seek tool of Excel® is employed for numerical solutions. Eqs. (31) and (34) gave two sets of solutions while Eqs. (37) and (38) gave four sets of solutions. Among these four sets of solutions two belong to the modes for the upper side of the base, while the other two are the symmetric solutions for the lower side of the base. The upper side assembly modes are depicted in Fig. (2), which is perfectly in accordance with the modes obtained in [1].

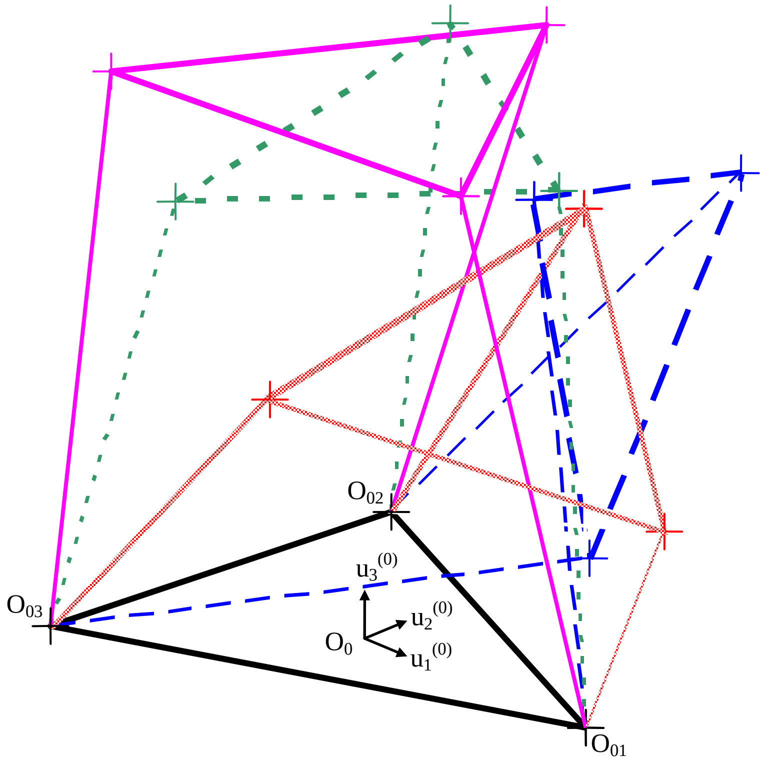

Figure 2: Upper side assembly modes for the non-parallel case for R = 100, r = 80, α_{12} = 130°, α_{13} = 220°, α_{45} = 100°, α_{46} = 260°, s_{14} = 190, s_{25} = 200, s_{36} = 210 units.

4.2 Parallel Mode

Substituting Eq. (25) in Eqs. (13-14) we get

$$\theta_{01}+\theta_{47}=\theta_{02}+\theta_{57}+\alpha_{12}-\alpha_{45}=\theta_{03}+\theta_{67}+\alpha_{13}-\alpha_{46} \tag{39}$$

In Eq. (25) we shall consider only the = 0 case and disregard the = π case, because either case results in the same orientation of the platform. By eqs. (18-19)

$$\left[\mathrm{R}e^{-\tilde{\mathbf{u}}_3\theta_{01}}\left(\mathbf{I}-e^{\tilde{\mathbf{u}}_3\alpha_{12}}\right)-\mathrm{r}e^{\tilde{\mathbf{u}}_3\theta_{47}}\left(\mathbf{I}-e^{\tilde{\mathbf{u}}_3\alpha_{45}}\right)\right]\mathbf{u}_1=\left[s_{25}e^{\tilde{\mathbf{u}}_3(\theta_{02}-\theta_{01}+\alpha_{12})}e^{\tilde{\mathbf{u}}_1\phi_{02}}-s_{14}e^{\tilde{\mathbf{u}}_1\phi_{01}}\right]\mathbf{u}_3 \tag{40}$$

$$\left[\mathrm{R}e^{-\tilde{\mathbf{u}}_3\theta_{01}}\left(\mathbf{I}-e^{\tilde{\mathbf{u}}_3\alpha_{13}}\right)-\mathrm{r}e^{\tilde{\mathbf{u}}_3\theta_{47}}\left(\mathbf{I}-e^{\tilde{\mathbf{u}}_3\alpha_{46}}\right)\right]\mathbf{u}_1=\left[s_{36}e^{\tilde{\mathbf{u}}_3(\theta_{03}-\theta_{01}+\alpha_{13})}e^{\tilde{\mathbf{u}}_1\phi_{03}}-s_{14}e^{\tilde{\mathbf{u}}_1\phi_{01}}\right]\mathbf{u}_3 \tag{41}$$

Eqs. (40-41) constitute 6 scalar equations in 7 angular unknowns, which means that one of these angular variables may be chosen to be free. Hence the PM has 4 dofs in the parallel mode and the platform motion is of Schönflies type. Let ϕ_{01} be the fourth input. Scalar components of Eqs. (40-41):

$$\left[c\theta_{01}-c\left(\theta_{01}-\alpha_{12}\right)\right]R-\left[c\theta_{47}-c\left(\theta_{47}-\alpha_{45}\right)\right]r=s\left(\theta_{02}-\theta_{01}+\alpha_{12}\right)s\phi_{02}s_{25} \quad (42)$$

$$\left[s\theta_{01}-s\left(\theta_{01}-\alpha_{12}\right)\right]R-\left[s\theta_{47}-s\left(\theta_{47}+\alpha_{45}\right)\right]r=c\left(\theta_{02}-\theta_{01}+\alpha_{12}\right)s\phi_{02}s_{25}-s\phi_{01}s_{14} \quad (43)$$

$$c\phi_{01}s_{14}=c\phi_{02}s_{25} \quad (44)$$

$$\left[c\theta_{01}-c\left(\theta_{01}-\alpha_{13}\right)\right]R-\left[c\theta_{47}-c\left(\theta_{47}-\alpha_{46}\right)\right]r=s\left(\theta_{03}-\theta_{01}+\alpha_{13}\right)s\phi_{03}s_{36} \quad (45)$$

$$\left[s\theta_{01}-s\left(\theta_{01}-\alpha_{13}\right)\right]R-\left[s\theta_{47}-s\left(\theta_{47}+\alpha_{46}\right)\right]r=c\left(\theta_{03}-\theta_{01}+\alpha_{13}\right)s\phi_{03}s_{36}-s\phi_{01}s_{14} \quad (46)$$

$$c\phi_{01}s_{14}=c\phi_{03}s_{36} \quad (47)$$

Eqs. (44) and (47) give the distance of P_p to P_b: $c\phi_{01}s_{14} = c\phi_{02}s_{25} = c\phi_{03}s_{36}$. ϕ_{02} and ϕ_{03} can be solved in terms of ϕ_{01} from these two equations (At most 4 pairs of ϕ_{02}, ϕ_{03} are obtained.). On the other hand, $\sigma_{14} = c\phi_{01}s_{14}$, $\sigma_{25} = c\phi_{02}s_{25}$, $\sigma_{36} = c\phi_{03}s_{36}$ represent the projections of the limbs onto the $\vec{\mathbf{u}}_1^{(0)}\ \vec{\mathbf{u}}_2^{(0)}$ plane and the problem becomes the direct kinematics solution of a 3-R<u>P</u>R planar PM with prismatic inputs σ_{14}, σ_{25}, σ_{36}. It is known that there exists at most 6 assembly modes for a 3-R<u>P</u>R planar PM (See ex. [7]), so maximum number parallel modes for our 3-UPU PM is 12 (6 symmetric pairs with respect to the base). The solution algorithm is basically as follows: First eliminate θ_{02} from Eqs. (42-43) by leaving alone and squaring up the last terms. Similary eliminate θ_{03} from Eqs. (45-46). Equations in terms of θ_{01} and θ_{47} are obtained and solved as how we solved Eqs. (37-38). Then solve for θ_{02} and θ_{03} from Eqs. (42-43) and (45-46) using atan2.

5 Identical Base and Platform Case

When R = r, $\alpha_{12} = \alpha_{45}$ and $\alpha_{13} = \alpha_{46}$ for the non-parallel mode Eqs. (31-36) become

$$\left(1-c\alpha_{12}\right)\left(c\theta_{01}-c\theta_{47}\right)-s\alpha_{12}\left(s\theta_{01}+s\theta_{47}\right)=0 \quad (48)$$

$$\left\{\left[s\theta_{01}-s\left(\theta_{01}-\alpha_{12}\right)\right]+c\left(\phi_{01}+\phi_{47}\right)\left[s\theta_{47}-s\left(\theta_{47}+\alpha_{12}\right)\right]\right\}R+s\phi_{01}s_{14}=s\phi_{02}s_{25} \quad (49)$$

$$c\phi_{01}s_{14}-s\left(\phi_{01}+\phi_{47}\right)\left[\left(1-c\alpha_{12}\right)s\theta_{47}-s\alpha_{12}c\theta_{47}\right]R=c\phi_{02}s_{25} \quad (50)$$

$$\left(1-c\alpha_{13}\right)\left(c\theta_{01}-c\theta_{47}\right)-s\alpha_{13}\left(s\theta_{01}+s\theta_{47}\right)=0 \quad (51)$$

$$\left\{\left[s\theta_{01}-s\left(\theta_{01}-\alpha_{13}\right)\right]+c\left(\phi_{01}+\phi_{47}\right)\left[s\theta_{47}-s\left(\theta_{47}+\alpha_{13}\right)\right]\right\}R+s\phi_{01}s_{14}=s\phi_{03}s_{36} \quad (52)$$

$$c\phi_{01}s_{14}-s\left(\phi_{01}+\phi_{47}\right)\left[\left(1-c\alpha_{13}\right)s\theta_{47}-s\alpha_{13}c\theta_{47}\right]R=c\phi_{03}s_{36} \quad (53)$$

From Eqs. (48), (51) and (27).

$$\theta_{01}+\theta_{47}=\theta_{02}+\theta_{57}=\theta_{03}+\theta_{67}=0 \quad (54)$$

The remaining angles are solved in terms of θ_{01} and θ_{47} from Eqs. (49-50) and (52-53) by following Steps 3-5 in Section 4.1. One of θ_{01} and θ_{47} remains arbitrary, hence the manipulator has 4 dofs for non-parallel mode of the the identical base and platform case. The parallel mode solution does not differ from the generic case.

6 Conclusions

Forward kinematic analysis of a new type of 3-UPU parallel manipulator is performed using exponential rotation matrices. Assembly modes are investigated via an example. We determined that the PM with generic dimensions is kinematotropic with 4 or 3 dofs depending on whether the platform is parallel to the base or not. It is shown that several assembly modes exist for both cases. For the special case, where the base and the platform are identical the PM is overconstrained with global 4 dof.

Although the very same problem is already solved in [1] using Study parametrization of motion, the analysis with the exponential matrices proved useful. Use of exponential matrices in writing the orientation and position equations made us able to easily pre-process the equations, hence get simplified and partially decoupled equations. An immediate advantage of using the vector approach is that we can extract the special geometric properties from the equations; however with the quaternion approach we had to impose the special conditions by geometric reasoning. Furthermore, the vector equations give important physical feedback: We have shown that in the non-parallel assembly modes the angles associated with the revolute joints connected to the base and platform are purely determined by the base and platform geometry and do not depend on the prismatic inputs. This is a special property of the PM we could not derive via the analysis with Study parameters.

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